2.3.14

THE

MATHEMATICAL GAZETTE.

EDITED BY

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ON THE CONNECTION BETWEEN THE INSCRIBED AND ESCRIBED CIRCLES OF A TRIANGLE.

THE *chief* object of this paper is to investigate a *method* of obtaining from any proposition connected with the inscribed circle of a plane triangle a corresponding proposition connected with one of the escribed circles of a plane triangle.

If we let a, b, c represent three lengths and A, B, C three angles, with the following limitations:

I. All six quantities must be positive,

II. $A+B+C=\pi$,

III. $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$,

we can regard a, b, c, A, B, C as the sides and angles of a plane triangle, and then by the methods of Algebra and Trigonometry we can deduce a number of algebraical and trigonometrical relations connecting them.

Let us remove the limitation I., keeping II. and III. unchanged. The six quantities can no longer be regarded, in the same sense, as the sides and angles of a plane triangle.

We may do this by putting $-A$ for A, $\pi-B$ for B, $\pi-C$ for C, a for a, $-b$ for b, $-c$ for c.

Then any general relation connecting a, b, c, A, B, C, the sides and angles of a plane triangle, will still hold good after these changes have been made.

The same set of substitutions may be arrived at in another way, as follows:

Consider a plane triangle ABC (Fig. 1). Let us keep the base a or BC fixed, and suppose the vertex A to gradually move off to infinity in any direction, *e.g.* along a fixed line EF, in the direction of the arrow, and to come back from infinity on the

other side, until the triangle assumes the position shown in Fig. 2.

Intermediate positions of the triangle can easily be drawn when necessary. Of course, when the vertex A is at infinity, the sides AB and AC will be parallel.

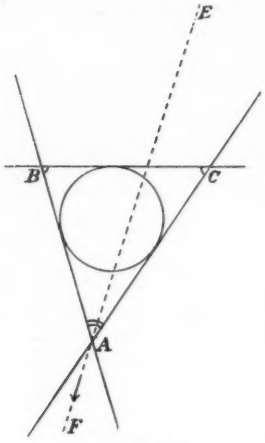


FIG. 1.

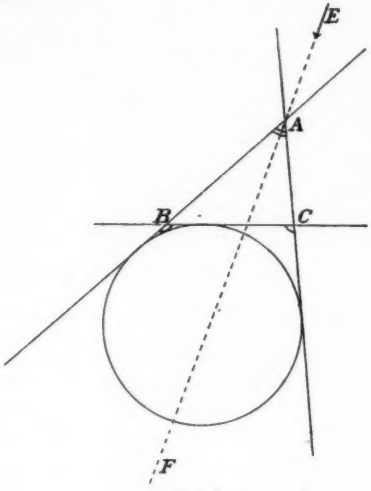


FIG. 2.

Now, by the law of continuity, any general property of a triangle ought to hold in *all* cases, however the triangle is varied.

Thus to any properties of Fig. 1 there will correspond properties of Fig. 2. But the *form* in which these corresponding properties are expressed will vary somewhat. When the vertex A passes through infinity, the angle A passes through a zero value and will change sign; b and c pass through infinite values and change sign; B and C become respectively $\pi-B$, $\pi-C$; a remains unaltered. Now these changes are exactly the same as those considered above.

By drawing a series of figures showing the intermediate positions of the triangle, it is generally easy to make the geometrical construction in Fig. 2 which corresponds to any geometrical construction in Fig. 1. The figures show that to any general property of the inscribed circle in Fig. 1 will correspond a property of one of the escribed circles in Fig. 2.

It is to be noted that, if we have any proposition given with a proof, the method we are considering will give not only a corresponding proposition but a corresponding proof also. Also the method applies equally well to geometrical constructions and proofs.

In reading questions connected with the inscribed circle one is often irritated by a small paragraph at the end, stating that a similar proposition holds with regard to the escribed circles. This similar proposition is often rather more complex than the original one, owing to the lack of symmetry, and is frequently somewhat difficult to state and prove.

The preceding method supplies an easy and direct method of at once getting the results required as well as a proof. Moreover, questions of the kind considered can *always* be stated in *pairs* instead of only occasionally as at present.

ILLUSTRATIONS OF THE METHOD.

NOTATION : $r, r_a, r_b, r_c, R, \Delta, s$ are used to denote respectively the radii of the inscribed, escribed, circumscribed circles, the area of the triangle, and the semiperimeter.

The substitutions suggested will change

$$r, r_a, r_b, r_c, s, s-a, s-b, s-c, \Delta, R$$

into $$r_a, r, -r_c, -r_b, a-s, -s, s-c, s-b, -\Delta, -R$$ respectively.

All the well-known formulae relating to the escribed circles of a triangle may be obtained at once, by means of this transformation, from the corresponding formulae relating to the inscribed circle, *e.g.*

(1) $$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

$$r_a = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

(2) $$\cos A+\cos B+\cos C = 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

$$\cos B+\cos C-\cos A = 4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-1.$$

(3) $$r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2},$$

$$r_a = s\tan\frac{A}{2} \quad = (s-c)\cot\frac{B}{2} = (s-b)\cot\frac{C}{2}.$$

(4) The sum of the diameters of the inscribed and circumscribed circles of any plane triangle is equal to $a\cot A+b\cot B+c\cot C$.

(*v.* Todhunter's *Trigonometry*, 8th ed., Chapter XVI., Ex. 18.)

Corresponding Theorem : The difference between the diameters of the circle escribed to the side BC and the circumcircle is

$$b\cot B+c\cot C-a\cot A.$$

(5) O is the centre of the circumcircle, P the orthocentre, G the centroid, and N the centre of the N.P.C., then

$$9OG^2 = 9R^2+8Rr+2r^2-2s^2,$$

$$OI^2 = R^2-2Rr,$$

$$IP^2 = 4R^2+4Rr+3r^2-s^2,$$

$$9IG^2 = 5r^2-16Rr+s^2.$$

Corresponding results are

$$9OG^2 = 9R^2 - 8Rr_a + 2r_a^2 - 2(s-a)^2,$$
$$OI_a^2 = R^2 + 2Rr_a,$$
$$I_aI^2 = 4R^2 - 4Rr_a + 3r_a^2 - (s-a)^2,$$
$$9I_aG^2 = 5r_a^2 + 16Rr_a + (s-a)^2.$$

(6) The distances of the orthocentre from the angular points are the roots of the equation

$$x^3 - 2(R+r)x^2 + (r^2 - 4R^2 + s^2)x - 2R\{s^2 - (r+2R)^2\} = 0.$$

(*v.* Hobson's *Plane Trig.*, p. 213.)

Corresponding Theorem: PA, $-PB$, $-PC$ are the roots of the equation

$$x^3 + 2(r_a - R)x^2 + \{r_a^2 - 4R^2 + (s-a)^2\}x + 2R\{(2R - r_a)^2 - (s-a)^2\} = 0.$$

(*v. Mathematics from the Educational Times*, Vol. LXVII., p. 65, Problem 13358. Of course, innumerable examples might be given.) H. B. BILLUPS.

NOTES ON ELEMENTARY DYNAMICS V.

RELATIVE MOTION.

THE idea of Relative Motion is one that presents a good deal of difficulty to the beginner in Mechanics. The following Notes are offered to readers of the *Gazette* as indicating a mode of presenting the subject which I have found to be helpful to learners. A clear grip of the working ideas, rather than a discussion of their logical basis, is what is here aimed at.

In the fable of the Eagle and the Wren, the Eagle soars as high as his strength permits; then the Wren, who has been perched on his back, flies up a little higher, and so attains to a greater height than the Eagle. This gives us in the first place an example of relative *displacement*. The Wren's flight is her displacement relative to the Eagle, and her total displacement is got by adding her relative displacement to the displacement of the Eagle. For simplicity we may suppose the final positions of Eagle and Wren to be vertically above the starting point.

Thus if we denote by h_E, h_W, and h_{EW} respectively the heights reached by the Eagle and by the Wren, and the height of the Wren above the Eagle, we have

$$h_W = h_E + h_{EW}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I.)}$$
$$h_{EW} = h_W - h_E, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II.)}$$
$$h_E = h_W - h_{EW}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(III.)}$$

These equivalent statements may be thus expressed in words:

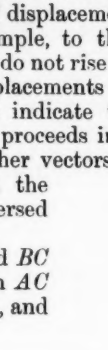
I. The displacement of W is got by compounding the displacement of W relative to E with the displacement of E.

II. The displacement of W relative to E is got by compounding the displacement of W with that of E reversed.

III. The displacement of E is got by compounding with the displacement of E that of W relatively to E with its sign reversed.

And it is easy to see that these equations and rules will apply to the *resolved parts in a given direction* of displacements which may have different directions; as, for example, to the heights attained by Eagle and Wren supposing they do not rise vertically.

The equations and rules apply also to displacements not in the same direction, if we take the sign + to indicate that what follows it is to be compounded with what proceeds in the same way as forces acting at a point (or any other vectors) are compounded; and the sign − to mean that the direction of what follows it is to be reversed before compounding.



Thus if AB be the flight of the Eagle, and BC that of the Wren relative to the Eagle, then AC will be the total displacement of the Wren, and we have

$$AC = AB + BC,$$
$$BC = AC - AB,$$
$$AB = AC - BC.$$

We can also extend the application of the preceding to *velocities* which are simply time-rates of change of displacement (since a velocity is represented by the displacement it would produce in unit time); to *accelerations*, which are time-rates of change of velocity; to *momenta*, and in fact to any vector quantities associated with the position or motion of bodies.

The general rules may be thus indicated:

I. $(P)+(P, Q)=(Q)$;
II. $(P, Q)=(Q)-(P)$;
III. $(P)=(Q)-(P, Q)$;

where (P) and (Q) denote the velocities (or accelerations, etc.) of points P and Q, and (P, Q) denotes the velocity (or acceleration, or etc.) of Q relative to P.

Take, for example, this question: A body P is thrown upwards with velocity $3g$, and two seconds afterwards a body Q is thrown upwards from the same point with velocity $2g$. Find when and where they will meet.

When Q starts, P has velocity $3g-2g=g$; therefore initial velocity of Q relative to P is $2g-g=g$. And at this instant the height P has attained is

$$= \text{mean vel.} \times \text{time of } P\text{'s motion} = \frac{3g+g}{2} \times 2 = 4g.$$

The acceleration of Q relative to P downwards is $g-g=0$.

Thus the question reduces to this: How long will a body with initial velocity g, and no acceleration, take to overtake a body distant $4g$ from it? The answer is obviously 4 seconds.

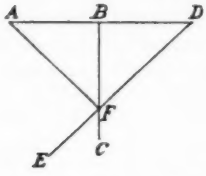
To get the *place* of meeting, consider the motion of Q relative to the ground. The velocity after 4 seconds is $2g-4g=-2g$.

Hence mean velocity = 0; therefore space passed over = 0. Thus the bodies P and Q meet at the place of starting.

Note that in studying the relative motion of two bodies, their initial relative position and velocity must be taken into account, as well as their relative acceleration.

As more general results of applying the method of Relative Motion to bodies falling freely under gravity, we may note that they have *no relative acceleration*; hence their *relative velocities* are uniform and fixed in direction. Hence if they are projected from the same point at the same time, their relative displacements (given by the lines joining them) being in the same direction as the relative velocities, are fixed in direction and increase or decrease uniformly. As an example of relative motion when the bodies move in different directions, take the following, which we shall solve geometrically and also analytically: A person travelling eastwards at the rate of 4 miles an hour finds that the wind seems to blow from the north: on doubling his speed, it seems to blow from the N.E. Find the direction and speed of the wind.

Draw AB eastwards = 4, and BC southwards in the direction of the relative velocity of the wind.



So far we do not know the magnitude of the latter, but by Rule I., the wind's velocity must be represented by a line joining A to some point in BC.

Next produce AB to D making AD double of AB and draw DE to S.W. in the direction of the second relative velocity of the wind. As in the former case the wind's velocity is represented by a line from A to some point in DE.

Combining these results we see that the required velocity must be given by AF, where F is the point of intersection of BC and DE.

The figure now shows the speed of the wind to be $4\sqrt{2}$ miles per hour, and its direction to be from the N.W.

ANALYTICAL SOLUTION.

Consider the following table:

Components.	Eastward.	Southward.
Wind's velocity, - - - -	X	Y
Man's velocity, Case 1, - - -	4	0
Wind's relative velocity, Case 1, -	0	P
Man's velocity, Case 2, - - -	8	0
Wind's relative velocity, Case 2, -	$-Q/\sqrt{2}$	$Q/\sqrt{2}$

Using Rule 1, we have $X = 4 + 0$, $\quad Y = 0 + P$,

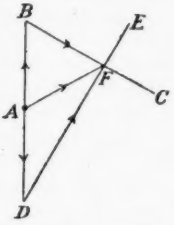
$$X = 8 - \frac{Q}{\sqrt{2}}, \quad Y = \frac{Q}{\sqrt{2}}.$$

Eliminating Q, we get $X = 4Y = 4$; the same result as before.

Another example: One steamer is steaming due north, with uniform speed, and another due south with equal speed. The smoke of the former forms a trail towards E.S.E., and that of the latter, a trail towards N.N.E. Find the direction of the wind, and its speed compared with that of the steamers.

[WOOLWICH, 1893.]

Consider that the smoke remains at rest relatively to the air, hence it shows the track of the steamer relative to the air, thus we know the direction of the velocity of the air relative to the steamer, which is equal and opposite to that of the steamer relative to the air. Thus the problem is really of the same type as the preceding one, and by inspecting the accompanying figure, we shall see at once that AB representing the velocity of the first, and AD that of the second steamer, that of the wind is given by AF, which is $= AB$, and makes an angle of $67\frac{1}{2}°$ with it.



A further example: A steamer starts eastward from a port, with uniform acceleration a, while a south wind blows with uniform velocity v. Find the shape of the trail of smoke left by the steamer. [WOOLWICH, 1893.]

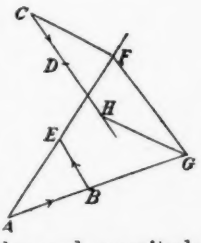
As in the preceding example, the trail of smoke marks the path of the steamer relative to the air. But the motion of the steamer relative to the air consists of an initial velocity v southward together with an acceleration a eastward. Comparing this with the well-known motion of projectiles, we see at once that the trail is a parabola whose axis is eastward, whose latus rectum is $= 2v^2/a$, and whose vertex is at that point of the air which started from port with the steamer. The parabolic trail of course floats northward with velocity v.

Given the positions and uniform velocities of two bodies P and Q, to find where and when they will be nearest to one another.

Let A, C be the initial positions of P, Q, and let AB, CD represent their velocities respectively.

Draw BE equal and parallel to CD, but in opposite sense. Then AE gives the velocity of P relative to Q.

Then, so far as motion relative to Q is concerned, Q is at rest, and P gets nearest to it at the point F which is the foot of the perpendicular from C upon AE. Thus CF gives the least distance from Q to P in direction and magnitude.



Obviously the time that elapses till this occurs is given by

the ratio $AF:AE$, which will be negative if F and E are on opposite sides of A.

To get the positions of P and Q at this time, draw FG parallel to BE or CD to meet AB in G, and complete the parallelogram $CFGH$.

Since $GA:BA=FA:EA$, it is clear that P gets to G at the time in question; and since GH is equal in magnitude and direction to FC, H must be the position of Q at that time. If AE produced passes through C, then P and Q strike one another.

A modified construction would enable us to find when P and Q were at a *given distance* from one another, or when their join was in a *given direction*. Instead of dropping the perpendicular C we should, in the former case, describe a circle with the given distance as radius and C as centre, cutting AE in F; and in the latter case draw a line from C in the given direction, so as to cut AE in F.

We may note that these results, except as to the *positions* of P and Q, would hold equally if P and Q were falling freely under gravity, with the given initial conditions; and the positions of P and Q could be got by the additional construction of lines vertically downward from G and H equal to the distance a stone would fall during the time in question.

R. F. MUIRHEAD.

MATHEMATICAL NOTES.

54. *Note on Section VI. of Professor Tanner's article on "A Class of Algebraic Functions"* (p. 153).

If W be a diaphoric function of $x, y, \ldots z$, and $\equiv U \times V$, then U and V must be diaphoric. For, let each argument be increased by δ, and let the new values of U and V contain respectively, *as highest powers of* δ, $U_r\delta^r$ and $V_s\delta^s$. Then, δ being an independent letter, $U_rV_s\delta^{r+s}$ must appear in the product. But the product is free from δ. Therefore $r=0$, $s=0$. R. W. GENESE.

55. *Euclid* IV. 12, 13, 14.

Let F, G, H, K, L be the angular points of an inscribed regular pentagon. At these points draw the tangents AB, BC, CD, DE, EA, and produce EA, CB to meet in P, and AB, DC to meet in Q.

Now (Euclid III. 26, 29) $LG=FH$, and $\angle$s PLG, PGL, QFH, QHF are equal to the exterior angle of a regular pentagon and therefore equal one another (Euclid III. 32). Hence the

$\triangle$s PLG, QFH are congruent, ...(a)

and $FQ=GP$.

But $BF=BG$.

Hence $BQ=BP$. .....(b)

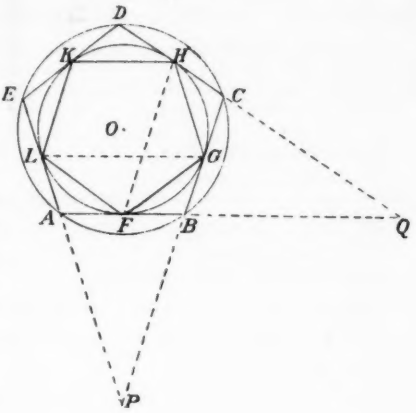
Again, $\angle ABP = \angle s\ BFG,\ BGF = \angle PGL$;

$\therefore\ AB \parallel GL$, and similarly $BC \parallel FH$,

hence (*a*), (*b*) [or Euc. I. 26], $\triangle$s ABP, BCQ are congruent, and $AB = BC$. Hence $ABCDE$ is equilateral and also equiangular (Euclid I. 15), therefore, etc.

Euclid IV. 13 may be derived at once from IV. 14. For the chords AB, BC, CD, DE, EA are proved equal, therefore they are equally distant from the centre (Euclid III. 14). Hence a circle with centre O (found in 14) and radius equal to the perpendicular distance of O from any one of the chords will touch the other chords and be inscribed in the pentagon.

R. TUCKER.



56. *Note on the Simson Line.*

I. If LMN is the Simson line of a point P on the circumcircle of a triangle ABC, each segment LM, MN, NL is proportional to the join of P to the angle containing the segment.

Let AD be the diameter through A.

Then, since the triangles PMN, BPC are similar, $\frac{MN}{PM} = \frac{BC}{PC}$; and since the triangles PMC, APD are similar, $\frac{PM}{PC} = \frac{AP}{AD}$;

$$\therefore\ MN = \frac{BC}{PC} \cdot PC \cdot \frac{AP}{AD} = AP \cdot \frac{BC}{2R} \propto AP;$$

similarly for LM, NL.

II. If P' be a point on the circle such that $PAD = P'AD$, and $N'L'M'$ be the Simson line of P', we have

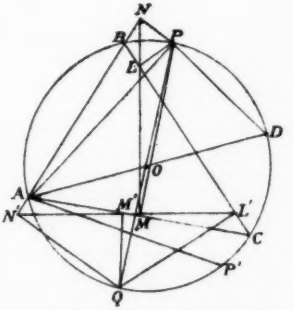
$$M'N' = AP' \cdot \frac{BC}{2R} = AP \cdot \frac{BC}{2R} = MN.$$

III. If $N'L'M'$ be the Simson line of Q the end of the diameter through P,

$$M'N'^2 = \frac{BC^2}{4R^2} AQ^2$$

$$= \frac{BC^2}{4R^2}(4R^2 - AP^2) = BC^2 - MN^2,$$

$$\therefore\ MN^2 + M'N'^2 = BC^2.$$



Hence the sum of the squares of the segments of the Simson lines of the extremities of a diameter is equal to the sum of the squares of the sides, *i.e.* is constant.

IV. If $MN=s_1$, $NL=s_2$, $LM=s_3$, and dashes refer to $L'M'N'$, then from the relations

$$s_1' \pm s_2' \pm s_3' = 0, \quad s_1^2 + s_1'^2 = a^2, \quad s_2^2 + s_2'^2 = b^2, \quad s_3^2 + s_3'^2 = c^2,$$

we get $$\sqrt{a^2 - s_1^2} \pm \sqrt{b^2 - s_2^2} \pm \sqrt{c^2 - s_3^2} = 0.$$

V. An easy method of drawing the Simson line of P is as follows:

AD the perpendicular on BC meets the circumcircle in G. PG cuts BC in E. Join P to the orthocentre H. Then the join of the mid-points of PE and PH is the Simson line of P.

W. J. GREENSTREET.

57. *Criteria of Divisibility.*

Let $\ldots abcdef$ be any number,

and $1, x, y, z, \ldots$ the successive remainders on dividing $10^0, 10^1, 10^2, 10^3, \ldots$ by any divisor F.

Then $(\ldots abcdef) \div F$ and $(f+ex+dy+cz+\ldots) \div F$ have the same remainder.

If the remainder of the latter is zero, F is a factor of $\ldots abcdef$.

Hence the following simple criteria:

Dividing $10^0, 10^1, 10^2, 10^3, \ldots$ by 3, 9 the remainders are 1, 1, ...

.....	6	..... 1, 4, 4, ...
.....	18	..... 1, 10, 4, 4, ...
.....	15, 45	..... 1, 10, 10, ...
.....	8	..... 1, 2, 4, 0, 0, ...
.....	125	..... 1, 10, 100, 0, 0 ....
.....	7	..... 1, 3, 2, 6, 4, 5, ...,
		or 1, 3, 2, −1, −3, −2, ...
.....	11	..... 1, 10, ...,
		or 1, −1, ...

Hence any number $\ldots abcdef$ is divisible by

3 or 9 if $f+ e+ d+ c+ b+ a+\ldots$	is $M(3)$ or $M(9)$,
6 if $f+ 4e+ 4d+ 4c+ 4b+ 4a+\ldots$	is $M(6)$,
15 or 45 if $f+10e+10d+10c+10b+10a+\ldots$	is $M(15)$ or $M(45)$,
18 if $f+10e+ 4d+ 4c+\ldots$	is $M(18)$,
8 if $f+ 2e+ 4d$	is $M(8)$,
125 if $f+10c+100d$	is $M(125)$,
7 if $f+ 3e+ 2d +6c+ 4b+ 5a+\ldots$	is $M(7)$,
or if $(f+3e+2d)-(c+3b+2a)+\ldots$	is $M(7)$,
11 if $f-e+d-c+b-a+\ldots$	is $M(11)$.

II. If $1, x, y, z, \ldots$ are the successive remainders on dividing

$$10^0, 10^n, 10^{2n}, 10^{3n}, \ldots \text{ by } F,$$

then any number is divisible by F if, on dividing the digits into groups (from right to left) of n digits, the sum of the products of the groups by $1, x, y, z, \ldots$ is a multiple of F.

(a) Take $n=2$. The remainders for 11 are 1, 1, ...;

$\therefore$ $\ldots abcdef$ is divisible by F if $(ef)+(cd)+(ab)+\ldots$ is $M(F)$.

[Ex. 13893; we have $93+38+1=132=M(11)$.]

(b) Take $n=3$. The remainders for 13 are 1, 12, ..., or 1, −1, ...;

$\therefore$ $\ldots abcdef$ is divisible by F if $def-abc+\ldots$ is $M(F)$.

[Ex. 132639520; we have $\begin{matrix}520-639\\+132\end{matrix}=652-639=13.$]

(*c*) Take $n=4$. The remainders for 17 are 1, 4, 16, 13, ..., or 1, 4, −1, −4, ...

[Ex. 171904 1921 3856; we have $\begin{matrix}3856-1904\\+7684-\quad 68\end{matrix}=11540-1972=9568$ and $9568=17\times562+14$. ∴ 14 is the remainder on dividing 1719...3856 by 17.]

III. More generally, if 1, x, y, z, ... are successive remainders on dividing 10°, 10^{m}, 10^{m+n}, 10^{m+n+p}, ... by F, then ...*abcdef* is $M(F)$ if

(first group of m digits)+(next group of n digits)
$\times x$+(next group of p digits)$\times y+\ldots=M(F)$.

(*a*) Take $m=5$, $n=4$, $p=0$.
Remainders for 19 are 1, 3, −1, −3, ...
Divide into groups of 5, 4, 5, 4, ... digits, and multiply in order by 1, 3, −1, −3.
Then the number is divisible by 19 if the sum thus found is $M(19)$.

[Ex. 5711430 4152 59337; we have

$$\begin{matrix}59337-11430\\+12456-\quad 171\end{matrix}=60192=19\times3168=M(19).$$]

(*b*) Take $m=5$, $n=3$, $p=3$.
Remainders for 23 are 1, 19, 2, 22, 4, 21,
or 1, −4, 2, −1, 4, −2, ...
Divide into successive groups of 5, 3, 3, ... digits, multiply in order by 1, −4, 2, and proceed as before.

(*c*) Take $m=4=n$, $p=3$, $q=3$.
Remainders for 29 are 1, −5, −4, 2, −1, 5, 4, −2, ...;
groups are of 4, 4, 3, 3, ... digits;
multipliers are 1, −5, −4, 2, −1, 5, 4, −2, ..., and so on.

A considerable saving of labour is effected by throwing out every multiple of the divisor.
Take for example III. (*a*):

5	7	1	1	4	3	0		4	1	5	2		5	9	3	3	7	
	0	0	0	0	1	1		0	3	1	4		0	2	3	1	8	
					1	1		0	1	2	4		0	0	4	1	8	
					1	1		0	1	0	5		0	0	0	3	8	
					1	1	|	0	1	0	5	|	0	0	0	0	0	

$315-11=304=M(19)$.

A large number of these tests has been collected by the Abbé Gelin. For further, but less useful, developments see *Mathesis*, 2nd Series XI., 1892, p. 67, and 1st Series III., p. 104. W. J. GREENSTREET.

58. *On the general proof of the rule for finding the product of two algebraic expressions.*

Although this rule is always quoted explicitly in proving the Binomial Theorem, and constantly used implicitly, I have never seen a complete proof in any Algebra, the usual course being, as in Chrystal, to refer to it as an easy generalisation of simple cases of distribution of products.

The proof has the merits of being easy, of offering several good specimens of reasoning from the form of an expression, and of forming in many ways a good starting point for ideas about algebraic *functions*, which, in spite of Chrystal, are apt to be neglected when so many 'rules' have to be learnt for examination purposes.

The form in which I generally get it (I do not *give* it as a 'rule' at all) is somewhat as follows :

The product $(a+\alpha)(b+\beta)$ can be written out as

$$\left.\begin{array}{l} a(b+\beta)+\alpha(b+\beta), \\ b(a+\alpha)+\beta(a+\alpha), \\ ab+a\beta+\alpha b+\alpha\beta. \end{array}\right\} \text{(From 1st principles and the meaning of the symbols employed.)}$$

The first shows that every term in the third contains either a or α, never both, but always one. Similarly for b and β.

Taking now $(x+a+\alpha)(b+\beta)$, and further examples ;

similar results are obtained, and the number of terms in the distributed product found from the number of terms in each factor.

Then taking $(z+c+\gamma)(x+a+\alpha)(b+\beta)$,

and writing it as

$$z\{(x+a+\alpha)(b+\beta)\}+c\{(x+a+\alpha)(b+\beta)\}+\gamma\{(x+a+\alpha)(b+\beta)\},$$

similar results follow for three factors, and so on.

The proof that the terms of a product of any number of expressions are found by taking the partial products *in every possible way* is, I believe, easiest as given in Clifford's *Common Sense of the Exact Sciences*, modified slightly to suit a more general form.

A reversal of the process indicated above gives an easy proof that the distributed product of any number of expressions takes the form mentioned.

J. Elliott.

59. *On the Circular Measure of Angles.*

Every teacher knows how hard it is to treat this part of trigonometry in a perfectly satisfactory way. It was, I believe, pointed out by Whewell that the difficulty lies in the phrase 'length of an arc.' The meaning of this is not fixed because no need of a definition is felt, and it is not surprising that it should be difficult to avoid looseness in arguing about a thing which has not been exactly defined. Now in geometry the comparison of magnitudes depends ultimately on superposition. The length of a straight line, that is, the number of units of length it contains, is ascertained by superposition of it and multiples of it upon the unit and its multiples. But it is impossible by this means to compare lines of different curvature since they cannot be superposed without bending, and there is no way of guaranteeing that the bending does not change the length until we know what is meant by the length.

We may define the length of an arc as being the limit of the perimeter of an inscribed open polygon whose sides are indefinitely diminished. This affords a satisfactory basis for argument if we prove that the perimeter has a definite limit independent of the law of formation of the polygon. But this early introduction of the idea of an integral may be avoided, and the measurement of arcs put off to a later stage, if we make use of the following known geometrical theorems :

Sectors of the same circle vary as their angles (Euclid VI. 33).

Circles are to each other in the duplicate ratio of their radii (Euclid XII. 2).

Hence if the unit angle is taken to be half that of a sector of unit radius and unit area, the area of any sector is $\frac{1}{2}r^2\theta$.

The formula $\sin\theta<\theta<\tan\theta$ follows by comparison of the sector with two triangles as in Hall and Knight's *Elementary Trigonometry* (p. 261).

The length of a circular arc may be proved to have a definite value, and this value may be found, by proving that the perimeter of an inscribed polygon (not including the bounding radii) is less than twice area of sector

divided by radius, and greater than the same expression multiplied by $\cos^2\alpha$, where 4α is the angle subtended at the centre by the greatest of the sides. (For radius × chord of an arc = twice area of quadrilateral formed by the bounding radii of the arc and the chords of the two halves of the arc.)

A. C. DIXON.

60. *Note on the formula* $\Gamma(n)=\operatorname{Lim}_{m=\infty} m^n \dfrac{1.2.3\ldots m}{n(n+1)\ldots(n+m)}$.

The degree of approximation of this formula may be discussed as follows: The exponential theorem shows that $e^x>1+x$, and that $e^x<\dfrac{1}{1-x}$ if $x<1$.

Thus $\quad e^{\frac{x}{m}}>1+\dfrac{x}{m}$, and if $x<m$, $e^{\frac{x}{m}}<1\Big/\left(1-\dfrac{x}{m}\right)$,

or $\quad e^x>\left(1+\dfrac{x}{m}\right)^m$, and if $x<m$, $e^x<\left(1-\dfrac{x}{m}\right)^{-m}$.

Thus $\quad \Gamma(n)<\displaystyle\int_0^\infty \left(1+\frac{x}{m}\right)^{-m} x^{n-1}dx$ and $>\displaystyle\int_0^m \left(1-\frac{x}{m}\right)^{m} x^{n-1}dx$,

i.e. $<m^nB(m-n, n)$ and $>m^nB(m+1, n)$.

Put $m+n+1$ for m in the first of these, and take m to be a positive integer, then

$$\Gamma(n)<(m+n+1)^n\frac{m!}{n(n+1)\ldots(n+m)}, \text{ and } >m^n\frac{m!}{n(n+1)\ldots(n+m)}.$$

Thus the error in $\log\Gamma(n)$ caused by taking either of these approximations is less than

$$n\log\left(1+\frac{n+1}{m}\right) \text{ or than } n(n+1)/m,$$

which diminishes without limit as m increases.

Let us write $f(m)$ for $m^n\dfrac{m!}{n(n+1)\ldots(n+m)}$. A closer approximation to $\Gamma(n)$ may be found as follows:

$$f(m+1)/f(m)=\left(\frac{m+1}{m}\right)^n\cdot\frac{m+1}{m+n+1}.$$

Now $\log(1+x)<\dfrac{x}{1+x}+\dfrac{x^2}{2(1+x)}$ or $x-\dfrac{x^2}{2(1+x)}$, but $>\dfrac{x}{1+x}+\dfrac{x^2}{2(1+x)^2}$,

and $\quad >x-\dfrac{x^2}{2}$, if $x<1$.

Thus $\log f(m+1)-\log f(m)$

$$<n\left\{\frac{1}{m+1}+\frac{1}{2m(m+1)}\right\}-\left\{\frac{n}{m+1}-\frac{n^2}{2(m+1)^2}\right\},$$

which $\quad <\dfrac{1}{2}(n^2+n)/m(m+1)$ or $\dfrac{n^2+n}{2}\left(\dfrac{1}{m}-\dfrac{1}{m+1}\right)$;

and $\log f(m+1)-\log f(m)$

$$>n\left\{\frac{1}{m+1}+\frac{1}{2(m+1)^2}\right\}-\left\{\frac{n}{m+1}-\frac{n^2}{2(m+1)(m+n+1)}\right\},$$

which $\quad >\dfrac{n^2+n}{2(m+1)(m+2)}$ or $\dfrac{n^2+n}{2}\left(\dfrac{1}{m+1}-\dfrac{1}{m+2}\right)$, if $n<1$.

Thus, by summation, since $f(\infty)=\Gamma(n)$,

$$\log\Gamma(n)-\log f(m)<\frac{n^2+n}{2m} \text{ and } >\frac{n^2+n}{2(m+1)}.$$

But $\log\left(1+\frac{n+1}{2m}\right)^n$ lies between these same limits, and hence when $n<1$ the logarithms of $\Gamma(n)$ and of $\left(m+\frac{n+1}{2}\right)^n \frac{m!}{n(n+1)\dots(n+m)}$ differ by less than $\frac{n^2+n}{2m^2}$.

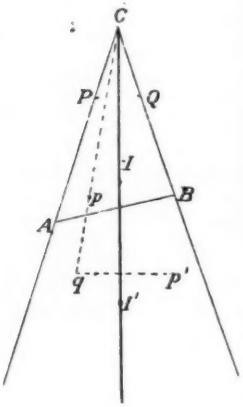
A. C. DIXON.

61. *A Transformation in Elementary Geometry.*

1. ABC is any triangle; CI the bisector of the vertical angle.

Let each point p in the plane of the triangle be transformed by the following rule into another point p':

Invert p from C, taking as radius of inversion a mean proportional to the sides CA, CB, and let this inverse be q. Invert q with respect to the line CI, that is, take its geometrical reflection p' in this line. This double inversion converts p into p'.



2. It is clear from the above that any straight line passing through C transforms into a straight line also through C; and that two straight lines which thus transform, the one into the other, make equal angles with the sides AC, BC. Again, that any circle becomes a straight line or circle according as it passes, or does not pass through C.

3. In particular it will be found that:

(i.) The circum-circle of ABC transforms into the base AB, and *vice versâ*.

(ii.) The in-circle (I) into a circle touching CA, CB, its chord of contact with the sides passing through the centre I' of the ex-circle (I').

(iii.) The ex-circle (I') into a circle touching CA, CB, its chord of contact passing through I.

4. As an application we have at once a theorem of Mannheim:

"Two circles touch CA, CB and the circum-circle, their chords of contact with these sides passing through I and I' respectively."

5. If we take as the square of the radius of inversion $\frac{1}{2}CA \cdot CB$,

A transforms into the middle point of BC,

B " " " CA.

The circumcentre becomes the foot of the perpendicular from C on AB. Thus the nine-points circle is changed into the circle through the ends of the base and circumcentre, and the ex-circle (I') becomes a circle touching CA, CB at P, Q, where PQ bisects CI at right angles.

This gives a solution of (230), since it is known that the nine-points circle touches (I').

C. E. M'VICKER.

EXAMINATION QUESTIONS AND PROBLEMS.

Our readers are earnestly asked to help in making this section of the GAZETTE *attractive by sending either original or selected problems.*

Solutions should be sent within three months of the date of publication. They should be written clearly on one side of the paper. Contractions not intended for printing should be avoided. Figures should be drawn with the greatest care on as small a scale as possible, and on a separate sheet.

The question need not be re-written, but the number should precede every solution.

The source of problems when not otherwise indicated is shown by —C. (Cambridge), O. (Oxford), D. (Dublin), W. (Woolwich), Sc. (Science and Art Department).

241. Squares $BCA''A'$, $CAB''B'$, $ABC''C'$ are described on the sides of any triangle ABC (Δ). AA', BB', CC' intersect in P, Q, R; AA'', etc., in P', Q', R'; $C'A''$, etc., in Π_1, Π_2, Π_3; $\Delta PQR = \Delta P'Q'R' = T$ (*v. E.T. Reprint*, vol. 58, p. 95);

$$\Delta\Pi_1\Pi_2\Pi_3 = S;$$

the triangles formed by the whole lines AA', etc., is T'; and by $C'A''$, etc., is S': $U = \Delta O_1O_2O_3$, where $O_1O_2O_3$ are the centres of the squares. Find T, T', S, S', U in terms of ω, the Brocard angle of Δ. Also show that PP', etc., intersect on the perpendiculars of Δ, and that the area V of the triangle formed by their intersections is given by $V(2+\cot\omega)^2 = \Delta(1+\cot\omega)^2$ or

$$V\Delta = (\Delta - T)^2.$$

W. S. COONEY.

242. Given $ABCD$, a convex quadrilateral, such that

$$AB+CD = BC+AD;$$

show geometrically that a circle may be inscribed to the quadrilateral.

R. F. DAVIS.

243. If $ABCDE$ is a pentagon with given sides on a fixed base AB, and having equal angles at C and E, the locus of D is a circle.

A. C. DIXON.

244. Through the centre of a rectangle $ABCD$ any straight line is drawn, meeting AB in P and CD in Q. R is any point on the circum-circle of the rectangle, and the perpendiculars from Q on AR and BR meet AD and BC in H and K respectively. Prove that HK passes through R and is perpendicular to PR.

W. J. DOBBS.

245. Given the perimeter of a triangle and the position of two sides. Find by elementary geometry the locus of a point dividing the base in a given ratio.

J. ELLIOTT.

246. If the sum of the plane angles at each vertex of a tetrahedron be two right angles, prove that the opposite edges are equal.

E. FENWICK.

247. CP, CP', CD, CD' are four straight lines. If (three letters denoting an area) $CPD = CP'D'$, $CPD' = CP'D$, and $CPP' = CDD'$; then the lines are pairs of conjugate radii of an ellipse. R. W. GENESE.

248. If P, Q, R be the Simson line of a point M on the circumcircle of the triangle ABC, and a, b, c the feet of the perpendiculars from M on the tangent at M, then the circles bcP, caQ, abR are respectively tangential to the sides of ABC.

W. J. GREENSTREET.

249. If the smallest prime factor of a number be not less than the cube root of that number, then the remaining factor is also prime. E. HILL.

250. If a circular cone be cut by a plane in an elliptic section, and if AB, AC are the greatest and least slant heights (from the vertex to the edge of the base), show that the volume of the cone so cut off is the G.M. of the right cones, cut off from the same cone, whose slant heights are respectively AB, AC.

A. LODGE.

251. AA', BB' are diameters of a rectangular hyperbola, and P is any point on the curve *within* the parallelogram $ABA'B'$. Prove that $PA \cdot PA' + PB \cdot PB' = AB \cdot AB'$. [As P passes along the curve to the outside, the sign of the term which vanishes *en route* must be changed.] C. E. M'VICKER.

252. Every whole number greater than 2 is the shortest side of an infinite number of rational plane triangles, whose sides are integers prime to each other. ARTEMAS MARTIN.

253. Find the hodographs of points on the rim or spokes of a wheel moving uniformly in a straight line, and of a point on the circumference of a circle rolling on a fixed circle.

R. F. MUIRHEAD

254. Walking from A to B, driving from B to C, and riding from C to A takes me $15\frac{1}{2}$ hours; driving from A to B, riding from B to C, and walking from C to A takes me 12 hours. Walking, the journey takes 22 hours, riding, $8\frac{1}{4}$, driving, 11 hours. To walk a mile, ride a mile, and drive a mile takes altogether $\frac{1}{2}$ hour. Find the rates at which I travel, and the distances A to B, etc. (Required a solution by Arithmetic.) T. ROACH.

255. O is the centre of curvature of a point P on an ellipse, centre C; the tangent at P meets the axes produced in T, t; and $CtQT$ is a rectangle. Show that (1) CO is perpendicular to PQ; (2) CO/PQ is the cotangent of the angle between CP and Tt.

E. P. ROUSE.

256. The base of a triangle is given in position and magnitude, and its vertex lies on a given straight line. If the sum of the sides is given, construct the triangle. J. V. THOMAS.

257. The points P, Q, R, S are taken on two orthogonal diameters of a circle, centre O, so that $OP=OQ=OR=OS$; AB, BC, CD, DA are the polars of the points. AP produced meets BR in p; prove that the locus of p and the analogous points is a cubical parabola, and that AP, BR and the analogous lines envelope hyperbolas. R. TUCKER.

258. Every number can be expressed as the sum of 1, 2, 3, or 4 squares. How many numbers are there less than 1000 which cannot be expressed as the sum of 1, 2, or 3 squares.
W. ALLEN WHITWORTH.

259. There is no number of six figures which does not contain factors prime to 111 111 1. W. P. WORKMAN.

SOLUTIONS.

A great number of solutions are in hand, and will be published as sufficient space is available.

Solutions are wanted for Nos. 90, 140, 150, 170, 171, 172, 194, 219 $\left(\text{for } \frac{c}{2} \text{ write } \frac{c}{a}\right)$, 236, 238.

ERRATA.

p. 166, line 26: for 'mq' read 'nq.'
p. 171, line 12: for 'ρ/m^3' read 'ρ_B/m^3.'
„ „ 13: for 'ρ' insert 'ρ_C.'

131. *Prove geometrically that* $\sin 3A = 4 \sin A \sin(60°+A) \sin(60°-A)$.
ST. JOHN'S (C.), Dec. 1894.

Solution by E. M. LANGLEY.

Let OX, OY be axes; $APQRE$ an arc of the unit circle.

$$A\hat{O}P = P\hat{O}Q = Q\hat{O}R = A. \quad A\hat{O}E = 60°.$$

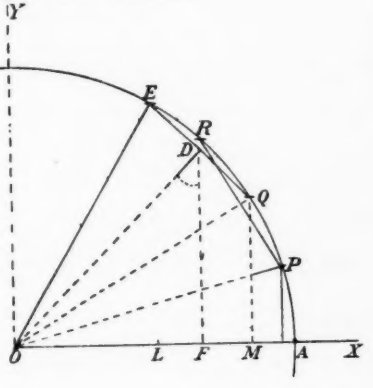
Let the bisector of $E\hat{O}Q$ meet EQ in D. Then

$A\hat{O}D = 30° + A$; $Q\hat{O}D = 30° - A$;

$O\hat{D}F = 60° - A$; $O\hat{Q}D = 60° + A$.

Draw EL, DF, QM perpendicular to OX. Then

$\sin 3A$ = projection of OR on OY
= sum of projections of OP, PR on OY
$= OP \sin A + PR \cos 2A$
$= 2OL \sin A + 2 \sin A \,.\, OM$
$= 4 \sin A \,.\, OF$
$= 4 \sin A \,.\, OD \sin O\hat{D}F$
$= 4 \sin A \sin(60° + A) \times \sin(60° - A)$.



Solution by R. F. DAVIS.

Let abc be the Simson line of S on the circum-circle of an equilateral triangle ABC. We know abc bisects OS, where O is the centre of the circle.

Let $S\hat{B}C=\theta$, and draw SR perpendicular to abc.

From the various cyclic quadrilaterals

$$SR=Sb\sin(60^\circ-\theta);\ Sb=SA\sin\theta;\ SA=2R\sin(60^\circ+\theta);$$

$$\therefore\ SR=2R\sin\theta\sin(60^\circ-\theta)\sin(60^\circ+\theta).$$

But $S\hat{O}C=2\theta$, and OC, Sc are parallel; $\therefore\ SR=\frac{1}{2}R\sin 3\theta$, whence, etc.

Solution by E. P. ROUSE.

CQD, DOC are equal circles, centres O, Q, diameter d. QO produced cuts the circles in X and Y. Any line parallel to OQ cutting the outer circumferences in P, R, is bisected by CD in N. Join PO and draw PF perpendicular to RO to cut XY in G. If $P\hat{R}O$ be A; $X\hat{O}P=O\hat{P}R=2A$; $P\hat{O}F=3A$; CXD is an equilateral triangle.

$$P\hat{O}C=120^\circ-2A;\ P\hat{D}C=60^\circ-A;\ P\hat{C}D=60^\circ+A;$$

$$\therefore\ PC \text{ and } PD \text{ are } d\sin(60^\circ-A) \text{ and } d\sin(60^\circ+A);$$

$$\frac{\sin 3A}{\sin A}=\frac{PR}{PO}=4\frac{PN}{d}=4\frac{PN.d}{d^2}=4\frac{PC.PD}{d^2}=4\sin(60^\circ-A)\sin(60^\circ+A).$$

The expression for $\cos 3A$ is proved in the same way, remembering that $PGYR$ is a parallelogram, and $OG=2pN$ where PpR cuts the circle (O) in P, p.

201, 208. Note by E. M. LANGLEY.

Take the bisectors of the angles between the diagonals as axes. One diagonal is

$$y=x\tan\frac{\omega}{2}. \qquad \text{.....(1)}$$

The circle is

$$x^2+y^2+2gx+2fy+c=0. \qquad \text{.....(2)}$$

The parabolas are

$$y^2+(2gx+2fy+c)\sin^2\frac{\omega}{2}=0, \qquad \text{.....(3)}$$

$$x^2+(2gx+2fy+c)\cos^2\frac{\omega}{2}=0. \qquad \text{.....(4)}$$

The product of the latera recta is $4fg\sin^2\frac{\omega}{2}\cos^2\frac{\omega}{2}=fg\sin^2\omega$.

Let $y=x\tan\frac{\omega}{2}$ cut the circle in (x_1, y_1), (x_2, y_2).

Then

$$x^2\sec^2\frac{\omega}{2}+2x\left(g+f\tan\frac{\omega}{2}\right)+c=0,$$

and

$$(x_1-x_2)^2=4\left\{\left(g+f\tan\frac{\omega}{2}\right)^2\cos^2\frac{\omega}{2}-c\right\}\cos^2\frac{\omega}{2},$$

$$d_1^2=4\left\{\left(g+f\tan\frac{\omega}{2}\right)^2\cos^2\frac{\omega}{2}-c\right\}.$$

Similarly

$$d_2^2=4\left\{\left(g-f\tan\frac{\omega}{2}\right)^2\cos^2\frac{\omega}{2}-c\right\}$$

$$\therefore\ d_1^2-d_2^2=8fg\sin^2\omega.$$

In No. 208, with the same axes as in 201, let $y=x\tan\frac{\omega}{2}$ pass through the centre. Then $f=g\tan\frac{\omega}{2}$.

The latus rectum of (4), above, is $2f\cos^2\frac{\omega}{2}$, and its axis is $x=-g\cos^2\frac{\omega}{2}$. Hence the intercept of axis is

$$2g\cos^2\frac{\omega}{2}\tan\frac{\omega}{2}=2f\cos^2\frac{\omega}{2}.$$

The equations (1)-(4) seem likely to be useful in problems of this class.

214. *A', B', C' are the mid-points of the triangle ABC. Any transversal through A meets $C'A'$ in Q, $A'B'$ in R; prove BQ, CR are parallel.*

G. RICHARDSON

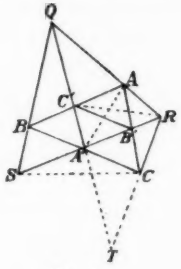


FIG. 1.

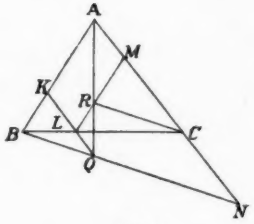


FIG. 2.

Solution by PROPOSER, W. E. JEFFARES, and REV. H. P. KNAPTON.

If $A'B'$ meet BQ in S, then, by similar triangles (fig. 1),

$$SA'/BC'=QA'/QC'=A'R/C'A\ ;$$

$$\therefore\ SA'=A'R.$$

Therefore the triangles $A'BS$, $A'CR$ are identical, and BS is equal and parallel to CR.

[Hence $BRCS$ is a parallelogram; similarly, if CR meets $C'A'$ in T, $BQCT$ is a parallelogram.]

Solution by R. F. DAVIS.

The triangles RAA', $RC'A'$ are equal (Euc. I. 37) (fig. 1).

Take away $RA'Q$ from each. Then $QAA'=QRC'$;

$$\therefore\ 2QAA'=2QCA'=QCB,\ \text{and}\ 2QRC'=QRB,$$

so that $QCB=QRB$; $\therefore$ etc.

Solution by C. SANDBERG.

Complete the parallelogram $RATC$. The angles BQA, RAT are equal, for BAQ, RTA are easily shown to be similar. Hence BQ is parallel to AT, *i.e.* to RC.

Generalization by J. BLAIKIE.

If through any point L in the base BC of a triangle ABC parallels be drawn to CA, AB meeting a transversal through A in Q and R, then BQ is parallel to CR.

Let KLQ, LRM be the parallels as in fig. 2.

Then $$\frac{AR}{RQ}=\frac{KL}{LQ}=\frac{AC}{CN};\quad\therefore\ RC \text{ is parallel to } BQ.$$

215. *If the diagonals of a convex quadrilateral are equal, and perpendiculars be drawn bisecting two opposite sides, these perpendiculars, measured from their point of intersection, are proportional to the sides on which they stand.*

(Amended from 183.)

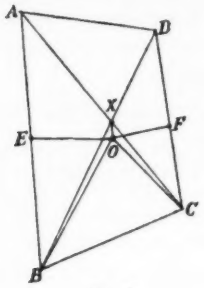


FIG. 1.

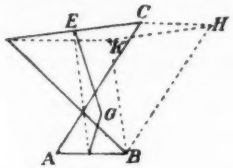


FIG. 2.

Solution by R. F. DAVIS.

From fig. 1 $AC=BD$, $OC=OD$, $OA=OB$; the figures $ABOX$, $DXOC$ are concyclic. If AC, BD intersect in X,

$$OCF=\pi-OXD=OXB=OAB,$$

and $$OF:CF=OE:AE\ ;\ \therefore \text{ etc.}$$

Solution by A. L. JONES.

Regard the equal diagonals as 'links,' or 'radius rods.' Then to move BD into the position AC, rotate round B to BH, and then parallel to itself into AC. (See fig. 2.)

If the perpendiculars meet in G, and K be the mid point of DH, EK is parallel to CH and half of it, *i.e.* is parallel to AB and equal to FB.

Also EF is perpendicular to DH, so that the similar triangles EGF, DCH give

$$\frac{EG}{GF}=\frac{DC}{CH}=\frac{DC}{AB}$$

Solution by C. SANDBERG.

OE, OF bisect AB, CD at right angles (fig. 1);

$$\therefore \quad BO=OA,\ OC=OD;$$

$$\therefore \quad \text{the triangles } AOC,\ BOD \text{ are equal;}$$

$$\therefore \quad \angle AOB=\angle DOC;$$

and as the triangles AOB, DOC are isosceles, therefore they are similar;

$$\therefore \quad OE:AB=OF:CD.$$

216. *If m edges meet at each vertex of a polyhedron of E edges, and each polygonal face have n sides, then $m^{-1}+n^{-1}=E^{-1}+2^{-1}$. Hence deduce that only five types of such polyhedra are possible, corresponding to the five regular solids.* R. F. MUIRHEAD.

Note by PROPOSER and others.

For solution, see Casey's *Spher. Trig.*, p. 128, § 121.

The form $E=\dfrac{2mn}{2(m+n)-mn}$ is not so convenient as $m^{-1}+n^{-1}=E^{-1}+2^{-1}$ for limiting the number of possible forms.

For $m \not< 3$, $n \not< 3$, and $m^{-1}+n^{-1} > \frac{1}{2}$. Thus $\frac{1}{3}+\frac{1}{3}$, $\frac{1}{3}+\frac{1}{4}$, $\frac{1}{3}+\frac{1}{5}$ give values of m, n ; 3, 3 ; 3, 4 ; 4, 3 ; 3, 5 ; 5, 3.

The cases $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$, $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$, making E infinite, give the three possible regular partitions of plane space, viz. : triangular, hexagonal, and square.

217. *Two similar and similarly-situated triangles are such that the same circle can be described in the one and about the other ; compare their areas, and prove that, if $A_1=4A_2$, the triangles are equilateral.* (W.)

Solution by G. HEPPEL.

If ma, mb, mc, $m^2\Delta$ be sides and area of A_1, a, b, c, Δ of A_2.

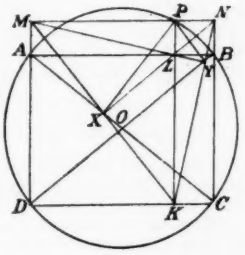
Then $$\frac{2m^2\Delta}{m\Sigma a}=\frac{abc}{4\Delta}, \text{ or } 8m\Delta^2=\Sigma a^2bc=4\Delta^2\Sigma \operatorname{cosec} B \operatorname{cosec} C ;$$

$$\therefore \quad m=\tfrac{1}{2}\Sigma \operatorname{cosec} B \operatorname{cosec} C, \text{ and areas are as } (\Sigma \operatorname{cosec} B \operatorname{cosec} C)^2 : 4.$$

If $A_1=4A_2$, $\Sigma \operatorname{cosec} B \operatorname{cosec} C=4$, which is the case when $A=B=C$.

218. *The projections on the four sides of a rectangle of any point on its circumcircle form an orthocentric system.*

E. M. LANGLEY.



Solution by W. J. GREENSTREET.

The figure explains itself.

XLN, *MLY*, *MXK* are the Simson lines of *P* for the triangles *ABC*, *ABD*, *ADC*. And in the triangle *MKN*, the three perpendiculars are *PK*, *NX*, *MY*, for *MAXLP*, *PLYBN* are concyclic ;

$\therefore$ *M*, *N*, *K*, *L* are an orthocentric system.

220. *The numbers* 1, 2, 3 ... $2n$ *are arranged round a circle at equal intervals. Through one of these, h, a diameter is drawn; for what values of n, h will the sum of the numbers on one side be equal to the sum of those on the other ?*

Pem., etc. (C.), 1896.

Solution by R. F. DAVIS, G. HEPPEL, and REV. H. P. KNAPTON.

Suppose $h<n$. The sum of all the numbers is $(2n+1)n$; excluding h and $h+n$, the semi-sum of the remainder is

$$\tfrac{1}{2}\{n(2n+1)-(n+2h)\}=n^2-h.$$

Thus $\overline{n-1}$ terms of $h+1, h+2, \ldots = n^2-h$.

This gives $$h=\frac{n+1}{2}.$$

Thus n must be odd, and the diameter required joins the numbers

$$\tfrac{1}{2}(n+1), \ \tfrac{1}{2}(3n+1).$$

221. *The opposite sides of a hexagon are parallel ; their extremities are joined and intersect in three points, the joins of which are parallel to the sides of the hexagon.*

J. BLAIKIE.

Solution by W. S. COONEY ; H. P. KNAPTON.

Let *ABCDEH* be the hexagon.

Let the joins of two of the parallel sides *BC*, *HE* meet at *P* and *Q*, and let *A'*, *B'*, etc., be feet of perpendiculars on *PQ*.

$$\frac{AA'}{EE'}=\frac{AA'}{CC'}\cdot\frac{CC'}{EE'}=\frac{AH}{CD}\cdot\frac{BC}{HE'}$$

So $$\frac{BB'}{DD'}=\frac{BB'}{HH'}\cdot\frac{HH'}{DD'}=\frac{BC}{HE}\cdot\frac{AH}{CD};\quad\therefore\quad\frac{AA'}{EE'}=\frac{BB'}{DD'},$$

and as ED is parallel to AB, PQ is easily proved parallel to ED; $\therefore$ etc.

222. *Prove that*

$$(2n)^2(r-2)mP+(r-4)^2(4n^3-n)/3=\sum_{k=1}^{k=2n-1}\{2mn(r-2)-k(r-4)\}^2$$

where P is any polygonal number, m its root, r its order;

e.g. $r=3,\ n=1,\qquad 8mT+1=(2m+1)^2$

$r=5,\ n=3,\ 648mP_5+35=(18m-1)^2+(18m-3)^2+(18m-5)^2.$

R. W. D. CHRISTIE.

Solution by H. W. LLOYD TANNER.

The factor m on the left of the equation and in each example should be suppressed.

The Σ from $k=1$ to $k=2n-1$ includes only the terms with odd values of k (1, 3, 5, ...), as indicated in the second example.

The right-hand side of the equation is

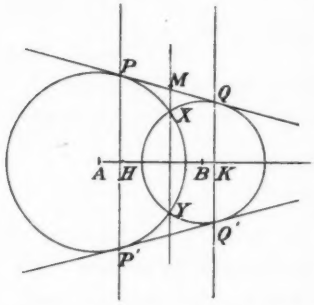
$$4m^2n^2(r-2)^2\Sigma 1-4mn(r-2)(r-4)\Sigma k+(r-4)^2\Sigma k^2$$

$$=4m^2n^2(r-2)^2n-4mn(r-2)(r-4)n^2+(r-4)^2(4n^3-n)/3,$$

and this agrees term for term with the left side (corrected) when $2P$ is replaced by its value $(r-2)m^2-(r-4)m$.

223. *Pairs of orthogonal circles pass through two fixed points; show that their common tangents envelop a fixed ellipse, and that the points of contact lie on two fixed lines.*

E. FENWICK.



Solution by C. E. M^{c}VICKER.

Let A, B be the centres of any pair of the circles, PQ, $P'Q'$ their common tangents, PP', QQ' cut AB in H, K. The join of the fixed points X, Y bisects PQ and therefore HK.

$$HK=PQ\cos\omega=c\cos^2\omega$$

where $\omega=(PQ, AB)$, the angle between PQ and AB. But $\sin\omega=(r-r')/c$.

Eliminating ω and using $r^2+r'^2=c^2$; $2rr'=c$. XY, we get $HK=XY$, *i.e.* HP, KQ are fixed lines. Again, the product of the perpendiculars from X, Y on $PQ=MX\,.\,MY\cos^2\omega$

$$=(\tfrac{1}{2}PQ)^2\cos^2\omega=(\tfrac{1}{2}HK)^2=(\tfrac{1}{2}XY)^2.$$

Hence PQ envelops a fixed ellipse, foci X, Y; minor axis $=XY$.

Solution by E. FENWICK.

With above lettering, we have XY bisects PQ and therefore HK. Also

$$AB\,.\,HK=PQ^2=AB^2-(AP-BQ)^2=AX^2+BX^2-(AX-BX)^2=2AX\,.\,BX;$$

$\therefore$ $AB\,.\,HK=AB\,.\,XY$ and $HK=XY=$a constant. The mid-point of HK is fixed; $\therefore$ PH, QK are fixed.

If p, p' be the perpendiculars from X, Y on PQ,

$$\frac{p}{MX}=\frac{p'}{MY}=\frac{HK}{PQ}\ ;\ \therefore\ \frac{pp'}{MX.MY}=\frac{HK^2}{PQ^2}\ ;\ \therefore\ pp'=\tfrac{1}{4}HK^2\ \because\ MX.MY=\tfrac{1}{4}HK^2\ ;$$

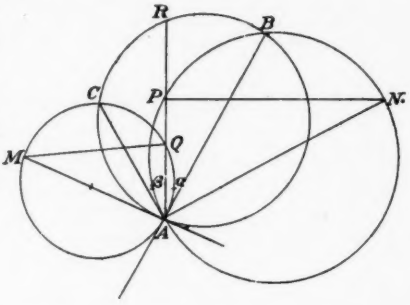
so that PQ touches a fixed ellipse, foci X, Y, eccentricity $\frac{1}{\sqrt{2}}$.

224. *APB, AQC are circles touching AC, AB respectively. AR any chord of the circum-circle of ABC cuts these circles in P and Q. Show that*

$$AR=AP+AQ.$$

R. W. Genese.

Solution by W. S. Cooney; H. G. Mayo.



Let d, d_1, d_2 be the diameters of AQC, APB, ABC. From circle ABC,

$$AR.BC=AB.CR+AC.BR;$$

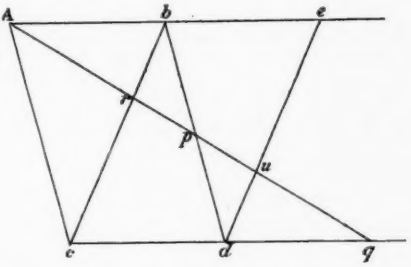
$$\therefore\ AR\,d\sin(\alpha+\beta)=AB.d\sin\beta+AC.d\sin\alpha.$$

But $AB=d_1\sin\alpha+\beta$,

$$AC=d_2\sin\alpha+\beta\ ;$$

$$\therefore\ AR=d_1\sin\beta+d_2\sin\alpha=AP+AQ.$$

Solution by R. F. Davis.



Let $Abdc$ (fig. 2) be a parallelogram, and let any transversal $Arpq$ cut bc, bd, cd respectively in r, p, q. Then take $be=Ab$; let de meet Aq in u. Then

$$\{Abe\infty\}=-1$$

and $d\{Abec\}$ is an H.R. Also

$$\{Apuq\}=-1\ ;$$

$\therefore$ Ap, Au, Aq are in H.P.

But $Au=2Ar$;

$$\therefore\ \frac{1}{Ar}=\frac{1}{Ap}+\frac{1}{Aq}.$$

Invert with respect to A, and the required relation follows.

Solution by J. V. Thomas.

From the similar triangles AQC, APB, $\frac{AQ}{AC}=\frac{PB}{AB}$ *i.e.* $\frac{AQ}{PB}=\frac{AC}{AB}$ .....(1)

Again, $C\hat{B}R=C\hat{A}R=A\hat{B}P$; $\therefore$ $P\hat{B}R=A\hat{B}C$, and as $P\hat{R}B=A\hat{C}B$,

the triangles PBR, ABC are similar. Hence $\frac{PR}{PB}=\frac{AC}{AB}$; .....(2)

$\therefore$ $AQ=PR$ by (1) and (2); $\therefore$ $AP+AQ=AP+PR=AR$.

227. *EF is the third diagonal of ABCD. H, K, L, M are points on AB, DC, BC, AD such that EH = AB, EK = CD, FL = BC, FM = DA. Then HKLM is a parallelogram.* E. M. LANGLEY, Jun.

Solution by W. S. COONEY.

Draw *KO* and *AO* parallel to *AB*, *EC* respectively. Join *OC*, *OM*. *HBOK*, *AOCD*, *FCOM*, *BOML* are evidently parallelograms;

$$\therefore \triangle HBL = \triangle KOM; \quad \therefore \quad HL = KM; \quad \therefore \text{ etc.}$$

Solution by J. L. THOMAS, E. Ll. TANNER, R. F. DAVIS, W. J. DOBBS.

Draw *DZ*, *BZ* parallel to *AB*, *AD* respectively. The triangles *EHK*, *DZC* are easily seen to be congruent, and *HK* is equal and parallel to *CZ*. So from the triangles *FML*, *FZC*, *HK* is equal and parallel to *ML*. Hence, etc.

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